Beard, R.E. 1959. Note on some mathematical mortality models. In: G.E.W.Wolstenholme and M.O'Connor (eds.).The Lifespan of Animals. Ciba Foundation Colloquium on Ageing. Little, Brown, Boston, 302-311

APPENDIX

NOTE ON SOME MATHEMATICAL MORTALITY MODELS

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1. A satisfying basis for a law of mortality would be a formula that, starting from some fundamental concepts about the biological ageing process, led to a distribution of deaths by age which was comparable with observational data. Such comparison would not be simple and straightforward because environmental and secular factors would introduce distortions as compared with the theoretical underlying distribution.

2. In the course of numerical work, extending over a number of years, on the expression of human mortality functions by mathematical formulae, various attempts have been made by the writer to develop an approach on this basis. The results obtained have not led to any satisfying formulae, but they are suggestive of different lines of approach and have been summarized below in the hope they may be of value to others interested in the subject. The note follows the sequence in which the ideas have developed in the mind of the writer and leads from considerations based on the force of mortality, μ_x, to those based on the curve of deaths, $\mu_x l_x$.

3. The first mathematical expression which provided a reasonable representation of the observed force of mortality in human data was that first proposed by Gompertz (1825) and later modified by Makeham (1867). Basically the "law" was derived by postulating a relationship between the rate of change of the force of mortality at any age and its value at that age. The next significant modification to the Makeham law was the system of curves devised by Perks in 1932 and of which the important formula was the logistic. Many human life-tables have been graduated by this basic curve, modified in some instances to allow for special features of the data, particularly at the younger and early middle ages, and the clear fact emerges that adult human mortality can be very well represented by a logistic curve of the form

$$\mu_x - A = B\,e^{\lambda x}/(1+D\,e^{\lambda x}) \qquad (1)$$

which will be referred to as a Perks curve since this is the name by which it is generally known by actuaries (Perks, 1932; Beard, 1936, 1939*a*, 1951*a*, 1952*a*; Registrar General, 1951; Mortality of Assured Lives, 1956).

4. Now μ_x is the ratio of the ordinate at age x of the curve of deaths to the area under the curve above age x. We may look upon the curve of deaths as a frequency distribution of deaths by age at death and for many types of frequency curves it will be found that this ratio has a sigmoid form. It is not apparent whether the satisfactory representation of μ_x by a Perks curve is because the formula has a theoretical significance or because the formula does provide a good approximation to the particular function of a family of frequency curves which can be used to represent the distribution of deaths by age (Perks, 1953).

5. What evidence is available tends to support the idea that the force of mortality does not continue to increase indefinitely with age. The concept of a limiting age by which all individuals must be dead (i.e. a maximum lifespan) does not seem to be in accordance with the facts—the use of a limiting age as a mathematical device to cut off a long slender tail has nothing to do with the present discussion. Formula (1) leads to an upper limit of B/D for μ_x and it is not without interest to note that the numerical values of B/D obtained from the graduation of human mortality data are of the same order as the force of mortality which can be deduced from select mortality tables as being appropriate to "damaged lives", i.e. about 0·57 (Beard, 1951*b*).

6. If the rapidly decreasing mortality associated with the infantile and growth period be ignored the pattern of human mortality then exhibits a basic sigmoid form on which are superimposed waves and other disturbances. The waves appear to be due largely to secular effects (e.g. selective effect of war deaths); the main disturbances are those arising from accidental deaths and the (rapidly disappearing) hump at the early adult ages from deaths from tuberculosis.

7. For a broad mathematical approach we will be concerned with (a) accidental deaths (assumed to be at a constant rate at all ages), (b) an upper limit to the rate of mortality, and (c) a progression in time.

Gompertz' law arises by using condition (c) only,

$$\text{i.e.}\quad d\mu_x/dx = \lambda\mu_x \qquad \text{whence} \quad \mu_x = B\,e^{\lambda x} \qquad (2)$$

Makeham's law arises by using conditions (a) and (c),

$$\text{i.e.}\quad d\mu_x/dx = \lambda(\mu_x - A) \qquad \text{whence} \quad \mu_x = A + B\,e^{\lambda x} \qquad (3)$$

Perks' law arises by using conditions (a), (b) and (c),

$$\text{i.e.}\quad d\mu_x/dx = \lambda(\mu_x - A)(E-\mu_x)/(E-A)$$

$$\text{whence}\quad \mu_x = A + \frac{(E-A)\,D\,e^{\lambda x}}{1+D\,e^{\lambda x}} \tag{4}$$

The Perks (logistic) relation can be expressed as stating that the rate of change of μ_x is proportional to the product of its value and the amount by which it falls short of its upper limiting value.

8. If the requirement of a constant upper limit for the rate of mortality is relaxed other formulae can be developed on similar lines to those of the preceding paragraph. For example,

$$\frac{d\mu_x}{dx} = \frac{\lambda(\mu_x - A)}{1+B(\mu_x - A)} \quad \text{gives} \quad w_x\,e^{w_x} = C\,e^{\lambda x}$$

$$\text{where}\quad w_x = B(\mu_x - A) \tag{5}$$

and

$$\frac{d\mu_x}{dx} = \frac{\lambda(\mu_x - A)\left(1+\frac{D}{B}\overline{\mu_x - A}\right)}{\left(1+\frac{2D}{B}\overline{\mu_x - A}\right)}$$

$$\text{gives}\quad \mu_x = A + \frac{B}{2D}\left(-1+\sqrt{1+4\,De^{\lambda x}}\right) \tag{6}$$

Formula (6) is equivalent to a continued fraction form for μ_x, i.e.

$$A + \cfrac{B\,e^{\lambda x}}{1+\cfrac{D\,e^{\lambda x}}{1+\dots}}$$

and the relationship between formulae (2) to (6) is clearly seen by expanding the expressions for μ_x in terms of powers of $e^{\lambda x}$, i.e.

formula (2) gives $B\,e^{\lambda x}$

" (3) " $A + B\,e^{\lambda x}$

" (4) " $A + B\,e^{\lambda x} - BD\,e^{2\lambda x} + BD^2\,e^{3\lambda x} - \dots$

" (5) " $A + B\,e^{\lambda x} - BD\,e^{2\lambda x} + \frac{3}{2}\,BD^2\,e^{3\lambda x} - \dots$

" (6) " $A + B\,e^{\lambda x} - BD\,e^{2\lambda x} + 2\,BD^2\,e^{3\lambda x} - \dots$

9. The differences between formulae (4), (5) and (6) will become apparent only at the old or very old ages and unless the data were extensive the differences would be unlikely to be significant for many numerical processes. From a scientific point of view the models are, of course, quite different.

10. An alternative approach to the question, but still based upon rates of mortality, is to determine the conditions necessary for μ_x to be a Perks (logistic) curve, given that the population can be stratified according to a longevity factor and that the basic mortality law is Makeham in form (Beard, 1952*b*). Thus let μ_k^s be the force of mortality at time ($\equiv$ age) k for the group having longevity factor s and let $\phi(s)\,ds$ be the proportion of the initial population having factor s. Then the survivors of $\phi(s)\,ds$ at time k are

$$\phi(s)\,ds\,.\,\exp\left(-\int_0^k \mu_t^s\,dt\right) \tag{7}$$

and the total survivors at time k

$$l_k = \int \phi(s)\exp\left(-\int_0^k \mu_t^s\,dt\right)ds \tag{8}$$

where the integral is taken over the whole range of s.
The force of mortality at time k ($= -d\log l_k/dk$) is then

$$\mu_k = \frac{\int \phi(s)\,\mu_k^s\exp\left(-\int_0^k \mu_t^s\,dt\right)ds}{\int \phi(s)\exp\left(-\int_0^k \mu_t^s\,dt\right)ds} \tag{9}$$

11. From formula (9) it will be noted that μ_k is a weighted mean of μ_k^s ($= \mu_k^{\bar{s}}$ say). Since the number of lives with heavier mortality will diminish more rapidly than those with lighter mortality, $\bar{s}$ will decrease with increasing k. If the basic mortality is Makeham in form, then $d\mu_k/dk$ will show a slackening off at the higher ages, i.e. the sigmoid feature shown by a logistic curve. In order to meet practical conditions some limitations are necessary on the form of $\phi(s)$; the lower limit must be $\geqslant 0$, but the upper limit can be ∞.

12. If it be assumed that $\phi(s)$ is a gamma function such that $\phi(s)\,ds = ks^p\,e^{-\gamma s}\,ds$ $(0 \leqslant s < \infty)$ and that the mortality function for $\phi(s)$ is $\mu_k^s = \alpha + \beta s\,e^{\lambda k}$, we have

$$\mu_k = \frac{\int_0^\infty ks^p\,e^{-\gamma s}(\alpha+\beta s\,e^{\lambda k})\exp\left(-\int_0^k (\alpha+\beta s\,e^{\lambda t})\,dt\right)ds}{\int_0^\infty ks^p\,e^{-\gamma s}\exp\left(-\int_0^k (\alpha+\beta s\,e^{\lambda t})\,dt\right)ds} \tag{10}$$

which reduces to

$$\mu_k = \alpha + \frac{(p+1)\beta\lambda\, e^{\lambda k}}{(\gamma\lambda-\beta)+\beta\, e^{\lambda k}} \tag{11}$$

which is a Perks (logistic) form.

13. The results of the immediately preceding paragraphs are interesting in that the limiting value of μ_k arises from the manner in which the "mixed" population runs off. They have a certain appeal in that they are based on the assumption that the population is not homogeneous in regard to a mortality (or longevity) factor and that the mortality for an individual group continues to increase indefinitely. The limiting value of $\mu_k - \alpha$ as $k \to \infty$ from formula (11) is $(p+1)\lambda = 4\lambda/\beta_1$ where β_1 is the Pearson moment function of $\phi(s)$. For human lives $\mu_x \sim 0{\cdot}6$ at the limit, according to one fairly recent mortality table, and $\lambda \sim 0{\cdot}1$ so that $\beta_1 \sim 0{\cdot}67$, i.e. a skew distribution with a tail towards the higher values of s. If s is a heredity factor, then stability of $\phi(s)$ over generations would imply fertility rates negatively correlated with longevity, otherwise the shorter reproductive period of those with higher values of s would lead to a falling average value of s in the population. It is an interesting coincidence that the distribution of married women according to number of children born has a β_1 coefficient of the order of $0{\cdot}7$ (Papers of Royal Commission on Population, 1950).

14. The assumption of other forms for $\phi(s)$ in formula (9) leads to other forms for μ_x which can have the appropriate shape but which are not convenient mathematically, and no experiments have been made in this direction.

15. From the point of view put forward in paragraph 1 formula (10) suffers from the objection that it is based on the assumption of a Makeham law, and is thus basically empirical. A further approach to the question is to build up models based on the so-called "shot hypothesis" in which individuals are assumed to be subject to random firings and are assumed to die when they have been "hit" a specified number of times. Two main types of model have been investigated, which are referred to below as the "forward" and "backward" models respectively. In the forward model hits are assumed to accumulate and death to occur when the total reaches a certain figure. In the backward model the individual is assumed to start with a quota of units which are progressively lost in time, death occurring when the total remaining falls below a certain figure.

16. The simplest forward model is derived by assuming that the chance that an individual is hit in an interval dt is p; this leads to a difference-differential equation

$$\frac{dl_t^\alpha}{dt} = -pl_t^\alpha + pl_t^{\alpha-1} \tag{12}$$

where l_t^α represents the number at time t who have been "hit" α times. If l_o is the number of individuals at time o then a solution of equation (12) is

$$l_t^\alpha = l_o\, e^{-pt}\,(pt)^\alpha/\alpha! \tag{13}$$

If the number of hits causing death is r, then the survivors at time t are

$$l_t = l_o\, e^{-pt}\{1+(pt)/1!+\ldots+(pt)^{r-1}/(r-1)!\}$$

and the deaths in the interval t to $t+dt$

$$\mu_t l_t = l_o\, e^{-pt}\, p^r t^{r-1}/(r-1)! \tag{14}$$

The force of mortality at time t is

$$\mu_t = \frac{p^r t^{r-1}}{(r-1)!}\bigg/\left\{1+\frac{pt}{1!}\ldots+\frac{(pt)^{r-1}}{(r-1)!}\right\}$$

$$= pe^{-pt}(pt)^{r-1}\bigg/\int_{pt}^{\infty} e^{-x}x^{r-1}\,dx \tag{15}$$

Formula (15) shows that the curve of deaths is an incomplete gamma function, or a Pearson type III curve. μ_t has the value 0 for $t=0$ and asymptotes to a value p at $t=\infty$ (Beard, 1959*b*).

17. A more natural function than μ_x in the present context is to use the function which bears the same relationship to $\mu_x l_x$ as μ_x does to l_x, i.e.

$$\frac{d(\log \mu_t l_t)}{dt} = \frac{1}{\mu_t}\frac{d\mu_t}{dt} - \mu_t$$

and from formula (14) we find this to be

$$\frac{d(\log \mu_t l_t)}{dt} = -p+\frac{r-1}{t} \tag{16}$$

18. Attempts to use the formula of paragraph 16 on human mortality data have been unsuccessful, the shape of $d(\log \mu_t l_t)/dt$ not

fitting well to observed values which show a negative second differential coefficient over the adult ages.

19. As an extension of formula (12) a model can be set up in which the "hits" in an interval can be single, double, etc., in known proportions. The basic relation then takes the form

$$\frac{dl_t^\alpha}{dt} = -pl_t^\alpha + p\sum_{r=1} f(r)\, l_t^{\alpha-r} \tag{17}$$

This can be integrated to

$$l_t^\alpha = e^{-pt}\int pe^{pt}\sum f(r)\, l_t^{\alpha-r}\, dt \tag{18}$$

and by noting that $l_t^0 = e^{-pt} l_0$ values of l_t^α can be obtained by successive integration. No experiments have been made using this form, mainly because the form of $d(\log \mu_t l_t)/dt$ seems to be unsuitable for human data. The form of $f(r)$ is also speculative.

20. A different forward model can be devised in which the probability of a "hit" is dependent on the number of "hits" recorded already. We then have the following

$$\frac{dl_t^\alpha}{dt} = -(\beta+p\alpha)\, l_t^\alpha + (\beta+p\cdot\overline{\alpha-1})\, l_t^{\alpha-1} \tag{19}$$

This can be integrated to give

$$l_t^\alpha = \frac{l_0\, e^{-(\beta+p\alpha)t}}{\alpha!}\left(\frac{\beta}{p}\right)\left(1+\frac{\beta}{p}\right)\cdots\left(\alpha-1+\frac{\beta}{p}\right)(e^{pt}-1)^\alpha \tag{20}$$

with

$$\frac{d(\log \mu_t l_t)}{dt} = -(\beta+\overline{\alpha-1}\, p) + \frac{(\alpha-1)\, p\, e^{pt}}{e^{pt}-1} \tag{21}$$

Here again the form of equation (21) does not accord with observations from human data.

21. In the attempts to fit these forward type formulae to human data it was found (Beard, 1950, 1952*c*) that satisfactory numerical results could be obtained by expressing $\mu_t l_t$ in gamma function form subject to a terminal age ω, i.e. the infinite tail of the curve is the opposite way round to what would be considered natural. This formula, after elimination of a constant element representing accidental mortality, can be derived from the difference-differential equation

$$\frac{dl_t^\alpha}{dt} = pl_t^\alpha - pl_t^{\alpha-1} \tag{22}$$

the solution of which leads to

$$l_t^\alpha = l_0\, e^{-p(\omega-t)}\{p(\omega-t)\}^\alpha/\alpha! \tag{23}$$

from which

$$\frac{d(\log \mu_t l_t)}{dt} = p - \frac{\alpha-1}{\omega-t}$$

if the deaths occur at the αth hit. In this formula $p \sim 0{\cdot}3$, $\alpha \sim 11$ and $\omega \sim 110$ for human mortality.

22. No obvious physical model applies to equation (22), but the relationship can be written in the backward form

$$\frac{dl_t^\alpha}{dt} = -\frac{\alpha}{\omega-t}\, l_t^\alpha + \frac{\alpha+1}{\omega-t}\, l_t^{\alpha+1} \tag{24}$$

in which the rate at which a unit is lost is proportional to the number of units remaining divided by the years of life remaining to the final age ω. From a biological point of view the concept of a final age by which the organism must be dead is unsatisfactory, but the fact that satisfactory numerical results arise only from a backward formula suggests that a closer study of this type of model might be more profitable.

23. The simplest backward model arises from the relationship

$$\frac{dl_t^\alpha}{dt} = -pl_t^\alpha + pl_t^{\alpha+1} \tag{25}$$

where the organism is assumed to lose a unit at rate p.
This has a solution

$$l_t^\alpha = l_0\, e^{-pt}\, (pt)^{n-\alpha}/(n-\alpha)! \tag{26}$$

where n is a maximum number of units. If death is assumed to occur when the number of units falls below r, we have

$$\frac{d(\log \mu_t l_t)}{dt} = -p + \frac{n-r}{t} \tag{27}$$

This is of similar form to equation (16) and is not suitable for human data.

24. By assuming that the rate of loss of a unit is proportional to the number of units remaining the relation

$$\frac{dl_t^\alpha}{dt} = -p(\beta+\alpha)\, l_t^\alpha + p(\beta+\alpha+1)\, l_t^{\alpha+1} \tag{28}$$

may be set up. This has the solution

$$l_t^\alpha = k\, e^{pt}/(1+D\,e^{pt})^{\beta+\alpha+1} \tag{29}$$

If death occurs when the units fall below α, we have

$$l_t = \sum_\alpha l_t^\alpha = k/(1+De^{pt})^{\beta+\alpha}$$

$$= l_0\, D(1+D)^{\beta+\alpha}/(1+De^{pt})^{\beta+\alpha} \tag{30}$$

We also have

$$\frac{d(\log \mu_t l_t)}{dt} = p - \frac{(\beta+\alpha+1)\, pDe^{p}}{1+De^{pt}} \tag{31}$$

and

$$\mu_t = \frac{p(\beta+\alpha)De^{pt}}{1+De^{pt}} \tag{32}$$

We have now found a difference equation model which leads to a Perks (logistic) formula for μ_t. In formula (31) the upper limit of μ_t is $p(\beta+\alpha)$; $p \sim 0{\cdot}1$ and the limit $\sim 0{\cdot}7$ so that $(\beta+\alpha) \sim 7$.

25. The distribution of α in the population at age 0 implied by equation (29) is a decreasing geometrical progression, i.e.

$$\frac{D}{1+D}, \frac{D}{(1+D)^2}, \ldots \frac{D}{(1+D)^\alpha}$$

For human mortality D is small (of the order of 10^{-5}) so that the distribution is very slowly decreasing with increasing α.

26. The significant result which emerges from the experiments made along these lines is that to provide results which have some reasonable semblance to observed human mortality the backward type of model has to be adopted. This is consistent with death being regarded as the culmination of a degenerative process such that death occurs when the organism reaches a certain level of degeneration. The mathematical models are based on numerical results for adult ages and interpolation back to birth is possibly a questionable process, a more suitable approach being to regard the life and death process as a period during which the organism is building up to a complex situation with a subsequent degeneration. This would lead to models in which the whole of life process would be looked upon as the resultant effect of two opposing forces.

27. Calculation of the moments of the distribution of deaths by age for a population of mice (Greenwood, 1928) shows that a Pearson type III (gamma function) would give a fair representation, but, as with the human data, the curve is the "opposite way round", i.e. subject to a terminal age. By inference the Perks (logistic) curve would give a fair representation of this data. No calculations have been made on animal data or on physical objects such as electric light bulbs and motor cars (e.g. Cramer, 1958) but it would seem worth while trying to find out if observed data of this latter type would distinguish between the two types of processes.

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